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Variable structure regulation of partially linearizable dynamics *

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Abstract: It is shown how the theory of decoupling and partial linearization of nonlinear affine systems can enhance the design of variable structure control systems and expand their range of applicability. Refined insights into the design of nonlinear switching surfaces and a new regular form are obtained. Application to an adjustable speed induction motor drive illustrates how this method allows stabilization of a periodic attractor.

Keywords: Variable structure control; exact linearization; zero dynamics; nonlinear systems; output regulation.

1. Introduction

Beginning with the paper of Byrnes and Isidori [3], several investigators including Byrnes and Isidori [4], Isidori and Moog [11] and van der Schaft [19] have articulated the notion of zero dynamics for a class of nonlinear systems of the form

$$\dot{x} = f(x) + G(x)u, \tag{1a}$$

$$y = h(x), \tag{1b}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$, with $m \leq n$, and f, $G = [g_1, \ldots, g_m]$ and h are smooth functions of x. In general terms, the notion of 'zero dynamics' correspond to the dynamics of the motion of (1) constrained to a manifold defined by $h(x) = 0$. The characterization of these constrained dynamics provides a convenient vehicle for decoupling and partial (input-output) linearization via nonlinear feedback.

The connection of this problem with variable structure control is immediately suggested because the constrained motion is analogous to the 'sliding motion' of variable structure control. When the concepts of variable structure control are combined with the ideas of partial linearization and zero dynamics for nonlinear dynamical systems we obtain an elegant characterization of control systems of this type. This perspective leads to several important observations and results. It will be seen that the 'equivalent control' of VS theory is precisely the feedback (partial) linearizing and stabilizing control. A convenient new regular form for VS control system design is obtained and the method of nonlinear switching surface design based on specification of the sliding dynamics in terms of differential equations is clarified and enhanced, resulting in a more systematic approach.

The combination of VS control with exact linearization has been previously discussed by Fernandez and Hedrick [6] and others, where the essential idea is to exploit linearizability (by smooth state feedback) in order to reduce the problem to one which is solvable by available methods. It is presumed that the output set is given and the associated zero dynamics are stable. However, unlike these earlier studies, the zero dynamics represent the central issue of interest herein. Our view is more in the spirit of Byrnes and Isidori [4] who prove that a sliding surface can be constructed for a minimum phase nonlinear system. However,

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the results herein provide a straightforward construction for the switching surface and do not depend on a local explicit representation of the zero dynamics. This last point is important in some applications where it is not intended, nor is it appropriate, for the zero dynamics to be characterized by a stable equilibrium point. We emphasize the significance of this issue with a practical example in which the zero dynamics evolve on a cylinder which contains a stable periodic orbit. In addition, our conclusions generalize the asymptotic linearization results of Bartolini and Tolezzi [1] to the multivariable case.

2. Sliding modes and output regulation: Preliminary remarks

Consider a nonlinear dynamical system of the form (1a). The controls u_i are discontinuous across smooth surfaces $s_i(x) = 0$, i.e.

$$u_i(x) = \begin{cases} u_i^+(x) & \text{if } s_i(x) > 0, \\ u_i^-(x) & \text{if } s_i(x) < 0, \end{cases} \quad i = 1, \dots, m, \tag{2}$$

and the control functions $u_i^+(x)$ and $u_i^-(x)$ are smooth functions of x.

The design of switching control systems of the type (1a), (2) is greatly facilitated by the deliberate introduction of sliding modes [18]. If there exists an open submanifold, M, of any intersection of discontinuity surfaces, $s_i(x) = 0$ for $i = 1, \dots, p \le m$, such that $s_i \dot{s}_i < 0$ in the neighborhood of almost every point in M, then it must be true that a trajectory once entering M remains in it until a boundary is reached. M is called a sliding manifold and the motion in M is called a sliding mode.

Variable structure control system design entails specification of the switching functions $s_i(x)$ and the control functions $u_i^+(x)$ and $u_i^-(x)$. It is typically a two step process which involves: (a) design of the 'sliding mode' dynamics by the choice of switching surfaces, and (b) design of the 'reaching' dynamics by the specification of the control functions.

If a trajectory of (1), (2) lies in a sliding manifold M, then it is characterized by the constraint $s(x) = 0$, and all time derivatives of $s(x)$ also vanish. The control is not defined by (2) when $s = 0$. Denote by u_{eq} (the equivalent control) the control which obtains while the trajectory remains in the manifold M. Then, u_{eq} is defined by

$$\dot{s} = S(x)\dot{x} = S(x)\{f(x) + G(x)u_{eq}\} := 0 \tag{3}$$

where $S(x) := (\partial/\partial x)s(x)$ and it is assumed that

$$\det\{S(x)G(x)\} \neq 0 \tag{4}$$

in which case we have

$$u_{eq} = -[S(x)G(x)]^{-1}S(x)f(x). \tag{5}$$

Motion in the sliding mode is then defined by

$$\dot{x} = \left[I - G(x)[S(x)G(x)]^{-1}S(x)\right]f(x), \quad s(x(0)) = 0. \tag{6}$$

It is easy to verify that trajectories which satisfy (6) and begin in a manifold defined by $s(x) = 0$, remain therein. If sliding occurs, it is characterized by (6). Conditions for the existence of such trajectories have been given by Utkin [18].

Consider the system (1) where y denotes a set of 'regulated' outputs. Our objective is to control the system so that $y(t) \to 0$ as $t \to \infty$. Thus, the desired ultimate system behavior corresponds to the condition defined by

$$h(x) = 0. \tag{7}$$

We would like to characterize the behavior defined by imposing the constraint (7) on the dynamics (1a). This problem is similar to that of identifying the sliding dynamics and we might proceed in like manner by setting the derivative of h equal to zero. A complication arises, however, because there is no reason to assume that $[\{(\partial/\partial x)h(x)\}G(x)]$ is not singular. Nevertheless, this problem has a known solution, obtained by successive differentiation of h, and which we will describe below. In this case, we obtain an 'effective' control $u_0(x)$ implied by the constraint (7), which is the natural counterpart of the equivalent control $u_{eq}(x)$ and a constrained dynamical system corresponding to (6).

3. Variable structure control design

In the following paragraphs, we will develop a view of variable structure control system design closely associated with methods of exact linearization [8] which has evolved from work of Krener [14], Brockett [2], Hirschorn [7] and Byrnes and Isidori [3]. We provide a sketch of the essentials, noting that the basic ideas for decoupling, partial linearization and stabilization of nonlinear systems are more fully developed in Isidori et al. [10], Charlet [5], Byrnes and Isidori [4] and Isidori [9].

Partial linearization and zero dynamics

Denote the k-th Lie (directional) derivative of the scalar function $\phi(x)$ with respect to the vector field $f(x)$ by $L_f^k(\phi)$. Now, by successive differentiation of the outputs y in (1b) we arrive at the following defintions. Let

$$r_i = \inf\left\{k \mid L_{g_j}\left(L_f^{k-1}(h_i)\right) \neq 0 \text{ for at least one } j\right\}. \tag{8}$$

Then r_i is the i-th characteristic number of (1). Let us define the column vector $\alpha(x)$ and the matrix $\rho(x)$:

$$\alpha_i(x) := L_f^{r_i}(h_i), \quad i = 1,\dots,m, \qquad \rho_{ij}(x) := L_{g_j}\left(L_f^{r_i-1}(h_i)\right), \quad i,\ j = 1,\dots,m. \tag{9}$$

Also define the vector $z \in R^r$, $r = r_1 + \cdots + r_m$, as

$$z := \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}, \quad z_i \in R^{r_i},\ i = 1,\dots,m, \tag{10a}$$

where

$$z_i^k = L_f^{k-1}(h_i), \quad k = 1,\dots,r_i \text{ and } i = 1,\dots,m. \tag{10b}$$

It is a straightforward calculation to verify that the variables z defined by (10) satisfy the relation

$$\dot{z} = Az + E\left[\alpha(x) + \rho(x)u\right], \tag{11a}$$

$$y = Cz, \tag{11b}$$

where the only nonzero rows of E are the m rows r_1, $r_1 + r_2, \dots, r$ and these form the identity I_m, the only nonzero columns of C are the columns 1, $r_1 + 1$, $r_1 + r_2 + 1, \dots, r - r_m + 1$ and these form the identity I_m, and

$$A = \operatorname{diag}(A_1,\dots,A_m), \quad A_i = \begin{bmatrix} 0 & I_{r_i-1} \\ 0 & 0 \end{bmatrix} \in R^{r_i \times r_i}. \tag{12}$$

Now we can easily establish the following result.

Proposition 1. *Suppose that $\rho(x)$ has continuous first derivatives with* $\det\{\rho(x)\} \neq 0$ *on* $M_0 = \{x \mid z(x) = 0\}$. *Then M_0 is a regular, $n - r$ dimensional submanifold of R^n and any trajectory segment $x(t)$, $t \in T$, T an open interval or R^1, which satisfies $h(x(t)) = 0$ on T lies entirely in M_0. Moreover, the control which obtains on T is*

$$u_0(x) = -\rho^{-1}(x)\alpha(x) \tag{13}$$

and every such trajectory segment with boundary condition $x(t_0) = x_0$, $t_0 \in T$, satisfies

$$\dot{x} = f(x) - G(x)\rho^{-1}(x)\alpha(x), \quad z(x(t_0)) = 0. \tag{14}$$

Proof. It follows from $\det\{\rho(x)\} \neq 0$ on M_0 that $\partial z(x)/\partial x$ is of maximum rank on the set $M_0 = \{x \mid z(x) = 0\}$ [9]. This maximal rank condition insures that M_0 is a well defined regular manifold of dimension $n - r$. From the definition of $z(x)$, it follows that y is identically zero on an open time interval if and only if z is zero on that interval. Thus, it follows from (11) that the unique control which must obtain during any motion constrained by $h(x) = 0$ is (13). With this control (1) reduces to (14). □

Note that the manifold M_0 defined by $z(x) = 0$ is invariant with respect to (14) so that any motion beginning in it remains therein. Indeed, (14) defines a flow on M_0 with all trajectories satisfying $y(t) = h(x(t)) = 0$. This justifies reference to (14) as the *zero output constrained dynamics* and to M_0 as the *zero dynamics manifold.*

It is assumed henceforth that the matrix $\rho(x)$ is nonsingular [1]. In this case, we can apply the feedback control law

$$u = -\rho^{-1}(x)[\alpha(x) - v] \tag{15}$$

where v is a new control input. Thus, we have the linearized input–output model

$$\dot{z} = Az + Ev, \tag{16a}$$

$$y = Cz. \tag{16b}$$

Note that the control law (15) simultaneously linearizes the input–output relation and decouples some of the dynamics (the zero dynamics) from the output.

It is not uncommon to refer to the variables z as the linearizable coordinates. The terminology of coordinates is justified by the maximal rank condition in the following way. Let $Z: R^n \to R^r$ denote the map realized as the function $z(x)$. By virtue of the maximal rank assumption and the implicit function theorem we can choose local coordinates $(y_1, \ldots, y_n)$ on R^n near any point $\alpha \in M_0$ such that $Z(y) = (y_1, \ldots, y_r)$. In terms of these coordinates M_0 is defined by $y_1 = 0, \ldots, y_r = 0$. As a matter of fact, the first r components correspond to the level sets $z(x) = c$ which exist for all c in some neighborhood of the origin in R^r. The remaining components $(y_{r+1}, \ldots, y_n)$ provide local coordinates on M_0.

Sliding dynamics

Let us proceed to design a variable structure controller for (1) by selecting a switching surface which is linear in z.

Proposition 2. *Let $s(x) = Kz(x)$ and suppose the conditions of Proposition 1 hold and $\partial s(x)/\partial x$ is of maximum rank on the set $M_s = \{x \mid s(x) = 0\}$. Then M_s is a regular $n - m$ dimensional submanifold of R^n which contains M_0. Moreover, if K is structured so that the m columns numbered r_1, $r_1 + r_2, \ldots, r$ compose an identity I_m, then for any trajectory segment $x(t)$, $t \in T$, T an open interval of R^1, which lies entirely in M_s, the control which obtains on T is*

$$u_{eq} = -\rho^{-1}(x)KAz - \rho^{-1}(x)\alpha(x) \tag{17}$$

[1] In the event that $\det\{\rho(x)\} = 0$, then further steps must be taken. See the 'zero dynamics algorithms' in Byrnes and Isidori [4] and the discussion in Fernandez and Hedrick [6].

and every such trajectory with boundary condition $x(t_0)=x_0 \in M_s$, $t_0 \in T$, *satisfies*

$$\dot{x}=f(x)-G(x)\rho^{-1}(x)\{\alpha(x)+KAz(x)\},\quad Kz(x(t_0))=0. \tag{18}$$

Proof. The maximum rank condition insures that M_s is a regular manifold of dimension $n-m$. M_0 is a submanifold of M_s in view of the definition of $s(x)$. Motion constrained by $s(x(t))=0$ must satisfy the sliding condition $\dot{s}=0$ and direct computation leads to (17) and (18). □

In this case observe that the manifold M_s is invariant with respect to the dynamics (18). The flow defined by (18) on M_s is called the *sliding dynamics* and the control defined by (17) is the *equivalent control*. Note that the equivalent control behaves as a linearizing feedback control. The partial state dynamics in sliding is obtained from (11a) and (17):

$$\dot{z}=[I-EK]Az,\quad Kz(t_0)=0. \tag{19}$$

Proposition 3. *Suppose the conditions of Propositions* 1 *and* 2 *apply. Then* M_0 *is an invariant manifold of the sliding dynamics* (18). *Moreover, if K is specified as*

$$K=\operatorname{diag}(k_1,\dots,k_m),\quad k_i=\left[a_{i1},\dots,a_{i,r_i-1},1\right], \tag{20}$$

where the m ordered sets of coefficients $\{a_{i1},\dots,a_{i,r_i-1}\}$, $i=1,\dots,m$, *each constitute a set of coefficients of a Hurwitz polynomial. Then every trajectory of* (18) *not beginning in* M_0 *approaches* M_0 *exponentially.*

Proof. Notice that (19) implies that the only trajectory of (18) with boundary condition $z(t_0)=0$ is $z(t)=0$ for all t and hence M_0 is an invariant set.

Note that $\operatorname{Im}[E]+\operatorname{Ker}[K]=R^r$ so that the motion of (19) can be conveniently divided into a motion in $\operatorname{Im}[E]$ and a motion in $\operatorname{Ker}[K]$ and the latter has eigenvalues which coincide with the transmission zeros of the triple (K, A, E); see Young et al. [20]. To prove that trajectories of (18) approach M_0 exponentially we need only show that all trajectories of (19) in $\operatorname{Ker}[K]$ approach the origin asymptotically. Let the matrix N be chosen so that its columns form a basis for $\operatorname{Ker}[K]$ and introduce the coordinate vectors $w \in R^{r-m}$ and $v \in R^m$, and write

$$z=Nw+Ev. \tag{21}$$

The inverse of (21) may be written

$$\begin{bmatrix} w \\ v \end{bmatrix}=\begin{bmatrix} M \\ K \end{bmatrix} z. \tag{22}$$

Direct calculation verifies that (19) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} w \\ v \end{bmatrix}=\begin{bmatrix} MAN & MAE \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix},\quad v(0)=0. \tag{23}$$

The result obtains if $\operatorname{Re}\lambda\{MAN\}<0$. If the matrix K is chosen in accordance with (20), then the eigenvalues of MAN are precisely the $r-m$ eigenvalues of the matrices

$$\begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots & \\ \vdots & \vdots & 0 & 1 & 0 & \\ \vdots & & & & 1 & \\ -a_{i1} & -a_{i2} & \dots & \dots & a_{ir_i-1} & \end{bmatrix},\quad i=1,\dots,m, \tag{24}$$

which lie in the open left half plane by assumption.

Reaching dynamics

The remaining step in VS control system design is the specification of the control functions $u_i^{\pm}$ such that the manifold $s(x)=0$ contains a stable submanifold which insures that sliding occurs. There are many ways of approaching the reaching design problem (Utkin [18]). We consider only one. Consider the positive definite quadratic form in s,

$$\mathscr{V} = s^{t}Qs. \tag{25}$$

A sliding mode exists on a submanifold of $s(x)=0$ which lies in a region of the state space on which the time rate of change of $\mathscr{V}$ is negative. Upon differentiation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V} = 2\dot{s}^{\mathrm{T}}Qs = 2[KAz+\alpha]^{\mathrm{T}}QKz + 2u^{\mathrm{T}}\rho^{\mathrm{T}}QKz. \tag{26}$$

If the controls are bounded, $|u_i| \leqslant \bar{U}_i > 0$, then obviously, to minimize the time rate of change of $\mathscr{V}$, we should choose

$$u_i(x) = -\bar{U}_i \operatorname{sign}(s_i^*), \quad i=1,\dots,m, \quad \text{and} \quad s^* = \rho^{\mathrm{T}}(x)QKz(x). \tag{27}$$

It follows that $\dot{\mathscr{V}}$ is negative provided

$$|\bar{U}^{\mathrm{T}}\rho^{\mathrm{T}}QKz| > |[KAz+\alpha]^{\mathrm{T}}QKz|. \tag{28}$$

A useful sufficient condition is that

$$|(\rho(x)\bar{U})_i| > |[KAz(x)+\alpha(x)]_i|. \tag{29}$$

Conditions (28) or (29) may be used to insure that the control bounds are of sufficient magnitude to guarantee sliding and to provide adequate reaching dynamics. This rather simple approach to reaching design is satisfactory when a 'bang-bang' control is acceptable.

$\mathscr{A} \subset M_0$ is a *stable attractor* of the zero dynamics if it is a closed invariant set and if for every neighborhood U of $\mathscr{A}$ in M_0 there is a neighborhood V of $\mathscr{A}$ in M_0 such that every trajectory of (19) beginning in V remains in U and tends to $\mathscr{A}$ as $t \to \infty$. The following proposition establishes conditions under which the variable structure controller (27) applied to (1) stabilizes $\mathscr{A}$ in R^n.

Proposition 4. *Suppose that the conditions of Propositions* 1, 2 and 3 *apply*; $\mathscr{D}$ *is an open region in* R^n *in which* (28) *is satisfied*; $\mathscr{D}_s = \mathscr{D} \cap M_s$ *is nonempty*; *and* $\mathscr{A} \subset M_0$ *is a bounded, stable attractor of the zero dynamics which is contained in* $\mathscr{D}_s \cap M_0$. *Then* $\mathscr{A}$ *is a stable attractor of the feedback system composed of* (1) *with feedback control law* (27).

Proof. Since $\mathscr{D}$ is an open region in R^n in which (28) is satisfied, a sliding mode exists in $\mathscr{D}_s = \mathscr{D} \cap M_s$ which is nonempty. In fact, $\mathscr{D}_0 = \mathscr{D}_s \cap M_0$ is also nonempty and it contains a bounded, stable attractor $\mathscr{A}$ of the zero dynamics (14). Proposition 3 implies that $\mathscr{A}$ is also a stable attractor of the sliding dynamics (18). Thus, for any neighborhood $\tilde{u}$ of $\mathscr{A}$ in M_s there is a neighborhood $\tilde{V}$ of $\mathscr{A}$ in M_s such that trajectories of (18) beginning in $\tilde{V}$ remain in $\tilde{U}$ and tend to $\mathscr{A}$ with increasing time. We must show that a similar property applies for neighborhoods of $\mathscr{A}$ in R^n with respect to the dynamics defined by (1) and (27). Let

$$\kappa_{\min} = \inf_{\mathscr{D}} \left\{ \bar{U}^{\mathrm{T}}\rho^{\mathrm{T}}QKz - [KAz+\alpha]^{\mathrm{T}}QKz \right\} > 0 \tag{30}$$

which exists by virtue of (28), and

$$\kappa_{\max} = \sup_{\mathscr{D}} \left\{ \left\| f(x) - \sum_{i=1}^{m} g_i(x)\bar{U}_i \operatorname{sign}(s_i^*) \right\|^2 \right\} < \infty \tag{31}$$

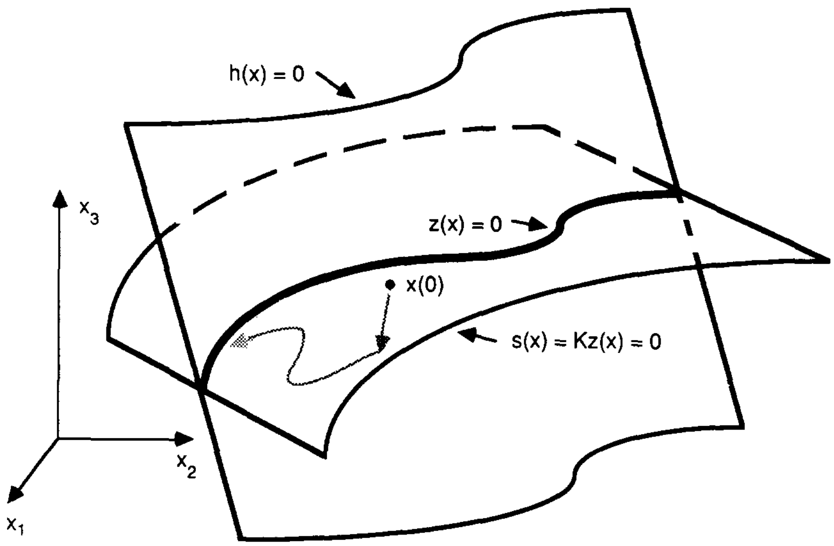

Fig. 1. The relationship between the output constraint manifold, the sliding manifold and the zero dynamics manifold in a three dimensional state space.

which exists because f and G are continuous and $\mathscr{D}$ is bounded, and where $\|\cdot\|$ denotes the Euclidean norm. Let $S(r, x_0)$ denote the open sphere in R^n of radius r and centered at x_0 and define the set

$$S(r) := \bigcup_{a \in \mathscr{A}} S(r, a). \tag{32}$$

Note that any element of $S(r)$ is at most a distance r from M_s and hence any trajectory starting in $S(r)$ will reach M_s in a finite time not greater than $t_r = r/\sqrt{\kappa_{\min}}$. Thus, any trajectory segment of the of the closed loop system beginning in $S(r)$ and terminating upon reaching M_s is entirely contained in the set $S(R)$ where

$$R = r\left\{1 + \sqrt{\kappa_{\max}/\kappa_{\min}}\right\} \tag{33}$$

Now, let $\hat{U}$ be any neighborhood of $\mathscr{A}$ in R^n. Define $\tilde{U} = \hat{U} \cap M_s$, so that $\tilde{U}$ is a neighborhood of $\mathscr{A}$ in M_s. Then there exists a neighborhood $\tilde{V}$ of $\mathscr{A}$ in M_s such that trajectories beginning in $\tilde{V}$ remain in $\tilde{U}$ and tend to $\mathscr{A}$ with increasing time. In view of (33), we can always choose r sufficiently small so that $S(R) \cap M_s \subset \tilde{V} \cap \mathscr{D}_s$. Then we identify $\hat{V} = S(r)$. It follows that trajectories of (1), (27) beginning in $\hat{V}$ remain in $\hat{U}$ and approach $\mathscr{A}$ as $t \to \infty$. □

Remarks on stability

First, let us denote $M_h = \{x \mid h(x) = 0\}$ and we assume that M_h is a regular submanifold of R^n of dimension $n - m$. Note that M_0 is a submanifold of both M_h and M_s so that M_0 lies in the intersection of M_h and M_s. The relationships between these manifolds are illustrated in Figure 1.

Our results imply that the closed loop system behaves as follows. If the initial state is sufficiently close to $\mathscr{D}_s$, the trajectory will eventually reach $\mathscr{D}_s$ and will thereafter approximate ideal sliding. Ideal sliding is characterized by (18) and sliding trajectories which remain in $\mathscr{D}_s$ approach $\mathscr{D}_0$ and eventually $\mathscr{A}$. That $\mathscr{A}$ is a stable attractor of (18) is obvious. However, this only implies that trajectories of (18) beginning sufficiently close to $\mathscr{A}$ approach $\mathscr{A}$. An important open problem is that of obtaining estimates of the domain of attraction. There are quite subtle issues here even in the simplest case where $\mathscr{A}$ is a globally stable equilibrium point of the zero dynamics (Kokotovic and Sussman [13]).

4. Example: The variable speed induction drive

In this section we give a brief account of the main issues of variable structure control system design for an adjustable speed induction motor. This problem has been previously considered by Izosimov and Utkin [12] and also Sabanovic and Izosimov [17]. A model for a round rotor, squirrel-cage motor with three stator phases and two rotor windings is:

$$M\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\omega\\ i_\mathrm{d}\\ i_\mathrm{q}\\ i_0\\ i_{\mathrm{f}1}\\ i_{\mathrm{f}2}\end{bmatrix} = -C\begin{bmatrix}\omega\\ i_\mathrm{d}\\ i_\mathrm{q}\\ i_0\\ i_{\mathrm{f}1}\\ i_{\mathrm{f}2}\end{bmatrix} + \begin{bmatrix}-\tau\\ v_\mathrm{d}\\ v_\mathrm{q}\\ v_0\\ 0\\ 0\end{bmatrix} \tag{34a}$$

with

$$M = \begin{bmatrix} J & 0 & 0 & 0 & 0 & 0\\ 0 & L_\mathrm{s} & 0 & 0 & L_\mathrm{fd} & 0\\ 0 & 0 & L_\mathrm{s} & 0 & 0 & L_\mathrm{fd}\\ 0 & 0 & 0 & L_0 & 0 & 0\\ 0 & L_\mathrm{fd} & 0 & 0 & L_\mathrm{f} & 0\\ 0 & 0 & L_\mathrm{fd} & 0 & 0 & L_\mathrm{f}\end{bmatrix}, \quad C = \begin{bmatrix} 0 & L_\mathrm{fd}i_{\mathrm{f}2} & -L_\mathrm{fd}i_{\mathrm{f}1} & 0 & 0 & 0\\ -L_\mathrm{fd}i_{\mathrm{f}2} & r & \omega L_\mathrm{s} & 0 & 0 & 0\\ L_\mathrm{df}i_{\mathrm{f}1} & -\omega L_\mathrm{s} & r & 0 & 0 & 0\\ 0 & 0 & 0 & r & 0 & 0\\ 0 & 0 & 0 & 0 & r_\mathrm{f} & 0\\ 0 & 0 & 0 & 0 & 0 & r_\mathrm{f}\end{bmatrix}, \tag{34b}$$

where the electrical torque is given by

$$T = L_\mathrm{fd}\{i_{\mathrm{f}1}i_\mathrm{q} - i_{\mathrm{f}2}i_\mathrm{d}\} \tag{35}$$

and the following nomenclature has been adopted:

ω	rotor angular velocity,	τ	mechanical torque (load),
v_f	field winding voltage,	T	electrical torque,
v_i, $i = 1, 2, 3$	stator winding voltages,	J	mechanical rotating inertia,
v_α, $\alpha = \mathrm{d, q}, 0$	stator Blondel voltages,	r, r_f	stator and field winding resistances,
$i_{\mathrm{f}i}$, $i = 1, 2$	field winding currents,	L_s	stator d&q axis inductances,
i_i, $i = 1, 2, 3$	stator winding currents,	L_0	stator zero sequence axis inductance,
i_α, $\alpha = \mathrm{d, q}, 0$	stator Blondel currents,	L_f	field winding self inductance,
		L_fd	field/stator mutual inductance.

If we use the notation $I^\mathrm{T} = [i_1, i_2, i_3]$, and $V^\mathrm{T} = [v_1, v_2, v_3]$ and $I_\mathrm{B}^\mathrm{T} = [i_\mathrm{d}, i_\mathrm{q}, i_0]$, $V_\mathrm{B}^\mathrm{T} = [v_\mathrm{d}, v_\mathrm{q}, v_0]$ then the (unitary, power conserving) Blondel transformation which relates stator winding currents and voltages I, V in the fixed reference frame to the Blondel current and voltages I_B, V_B in the frame attached to the rotor is

$$I_\mathrm{B} = BI, \qquad V_\mathrm{B} = BV, \tag{36}$$

$$B = \sqrt{\tfrac{2}{3}}\begin{bmatrix}\cos\theta & \cos(\theta - \frac{2}{3}\pi) & \cos(\theta + \frac{2}{3}\pi)\\ \sin\theta & \sin(\theta - \frac{2}{3}\pi) & \sin(\theta + \frac{2}{3}\pi)\\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}. \tag{37}$$

Notice that the induction motor has three control variables, the three stator phase voltages. Our objective is to design a feedback controller to regulate speed and two other quantities. In steady operation we desire balanced operation which may be defined to mean either $i_0 = 0$ or $v_0 = 0$. It is convenient to introduce a new state variable χ according to the relation

$$\dot{\chi} = v_0 \tag{38}$$

and to regulate χ. Of course $\chi = 0$ implies that v_0 vanishes almost everywhere. An appropriate choice for the third regulated variable is the rotor electromagnetic field magnitude, ψ, which is to maintain a specified constant value, ψ_0. Such an approach is advocated by Sabanovic and Izosimov [17] and Izosimov and Utkin [12]. They recommend that a sliding surface be specified so that on that surface

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi - \psi_0) + b(\psi - \psi_0) = 0, \quad b > 0, \tag{39a}$$

where

$$\psi = \left[\psi_1^2 + \psi_2^2\right]^{1/2}, \qquad \psi_1 = L_{\mathrm{fd}} i_{\mathrm{d}} + L_{\mathrm{f}} i_{\mathrm{f1}}, \quad \psi_2 = L_{\mathrm{fd}} i_{\mathrm{q}} + L_{\mathrm{f}} i_{\mathrm{f2}}. \tag{39b}$$

Thus, the design described in the above references is based on switching surfaces

$$s_1 := \frac{1}{J}\{-L_{\mathrm{fd}} i_{\mathrm{f}} i_{\mathrm{q}} - \tau\} + c(\omega - \omega_0) = \dot{\omega} + c(\omega - \omega_0), \quad c > 0, \tag{40a}$$

$$s_2 := \chi, \tag{40b}$$

$$s_3 := -r_{\mathrm{f}}\left[\psi_1^2 + \psi_2^2\right]^{-1/2}\{\psi_1 i_{\mathrm{f1}} + \psi_2 i_{\mathrm{f2}}\} - b(\psi - \psi_0) = \dot{\psi} + b(\psi - \psi_0), \quad b > 0. \tag{40c}$$

We will modify this strategy somewhat to illustrate the flexibility of the approach advocated in this paper. Let us specify the regulated outputs

$$y_1 = h_1(\omega, i_{\mathrm{d}}, i_{\mathrm{q}}, i_0, i_{\mathrm{f1}}, i_{\mathrm{f2}}, \chi) := \omega - \omega_0, \tag{41a}$$

$$y_2 = h_2(\omega, i_{\mathrm{d}}, i_{\mathrm{q}}, i_0, i_{\mathrm{f1}}, i_{\mathrm{f2}}, \chi) := \chi, \tag{41b}$$

$$y_3 = h_3(\omega, i_{\mathrm{d}}, i_{\mathrm{q}}, i_0, i_{\mathrm{f1}}, i_{\mathrm{f2}}, \chi) := \psi - \psi_0. \tag{41c}$$

Straightforward calculation leads to the conclusion that the characteristic numbers associated with the three outputs defined in (41) are, respectively, $r_1 = 2$, $r_2 = 1$, $r_3 = 2$. It follows that z is of dimension 5 and the zero dynamics are of dimension 2. In fact, we obtain by direct computation using the construction of Section 3,

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} \omega - \omega_0 \\ -L_{\mathrm{fd}} i_{\mathrm{f2}} i_{\mathrm{d}} + L_{\mathrm{fd}} i_{\mathrm{f1}} i_{\mathrm{q}} - \tau \\ \chi \\ \psi - \psi_0 \\ -r_{\mathrm{f}} \psi_1 i_{\mathrm{f1}} - r_{\mathrm{f}} \psi_2 i_{\mathrm{f2}} \end{bmatrix}. \tag{42}$$

Note that we can identify $z_1 = y_1$, $z_2 = \dot{y}_1$, $z_3 = y_2$, $z_4 = y_3$ and $z_5 = \dot{y}_3$. The corresponding differential equations of the linearizable dynamics, in the standard form of Section 3, Eq. (11), are

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_1 + \rho_{11} v_{\mathrm{d}} + \rho_{12} v_{\mathrm{q}} \\ v_0 \\ 0 \\ \alpha_3 + \rho_{31} v_{\mathrm{d}} + \rho_{32} v_{\mathrm{q}} \end{bmatrix} \tag{43}$$

where

$$\begin{bmatrix}\alpha_1\\ \alpha_3\end{bmatrix}=\begin{bmatrix}\dfrac{L_{fd}}{J}\left[-i_{f2}\; i_{f1}\; i_q\; -i_d\right]\\ -2r_f\left[L_{fd}i_{f1}\; L_{fd}i_{f2}\; L_{fd}i_d+2L_f i_{f1}\; L_{fd}i_q+2L_f i_{f2}\right]\end{bmatrix}$$

$$\cdot\begin{bmatrix}L_s & 0 & L_{fd} & 0\\ 0 & L_s & 0 & L_{fd}\\ L_{fd} & 0 & L_f & 0\\ 0 & L_{fd} & 0 & L_f\end{bmatrix}^{-1}\begin{bmatrix}\omega L_{fd}i_{f2}-ri_d-\omega L_s i_q\\ -\omega L_{fd}i_{f1}+\omega L_s i_d-ri_q\\ -r_f i_{f1}\\ -r_f i_{f2}\end{bmatrix},$$

$$\begin{bmatrix}\rho_{11} & \rho_{12}\\ \rho_{31} & \rho_{32}\end{bmatrix}=\begin{bmatrix}\dfrac{L_{fd}}{J}\left[-i_{f2}\; i_{f1}\; i_q\; -i_d\right]\\ -2r_f\left[L_{fd}i_{f1}\; L_{fd}i_{f2}\; L_{fd}i_d+2L_f i_{f1}\; L_{fd}i_q+2L_f i_{f2}\right]\end{bmatrix}$$

$$\cdot\begin{bmatrix}L_s & 0 & L_{fd} & 0\\ 0 & L_s & 0 & L_{fd}\\ L_{fd} & 0 & L_f & 0\\ 0 & L_{fd} & 0 & L_f\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}.$$

Now, let us characterize the zero dynamics. The zero dynamics evolve on the manifold defined by $z(x)=0$ so that $z_1=0$ implies $\omega=\omega_0$ and $z_3=\chi=0$ implies $v_0=0$ almost everywhere. Hence we can reduce (34) to

$$\frac{d}{dt}i_0=-\frac{r}{L_0}i_0, \tag{44a}$$

$$\begin{bmatrix}L_s & 0 & L_{fd} & 0\\ 0 & L_s & 0 & L_{fd}\\ L_{fd} & 0 & L_f & 0\\ 0 & L_{fd} & 0 & L_f\end{bmatrix}\frac{d}{dt}\begin{bmatrix}i_d\\ i_q\\ i_{f1}\\ i_{f2}\end{bmatrix}=-\begin{bmatrix}r & \omega_0 L_s & 0 & -\omega_0 L_{fd}\\ -\omega_0 L_s & r & \omega_0 L_{fd} & 0\\ 0 & 0 & r_f & 0\\ 0 & 0 & 0 & r_f\end{bmatrix}\begin{bmatrix}i_d\\ i_q\\ i_{f1}\\ i_{f2}\end{bmatrix}+\begin{bmatrix}v_d\\ v_q\\ 0\\ 0\end{bmatrix}. \tag{44b}$$

Note that since ω_0 is a constant, Eq. (48b) is linear. This fact can be exploited. We will first characterize solutions which satisfy the remaining constraints $z_2=0$, $z_4=0$, and $z_5=0$.

Note that by using the definitions of ψ_1, ψ_2 to eliminate the currents i_d, i_q the constraints $z_2=0$ and $z_5=0$ may be written

$$-\psi_1 i_{f2}+\psi_2 i_{f1}=\tau,$$

$$\psi_1 i_{f1}+\psi_2 i_{f2}=0.$$

From these equations we obtain the relation

$$\begin{bmatrix}i_{f1}\\ i_{f2}\end{bmatrix}=\frac{1}{\psi_0}\begin{bmatrix}0 & \tau\\ -\tau & 0\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \tag{45}$$

where we have also invoked the output constraint $z_4=0$, i.e.,

$$\psi_1^2+\psi_2^2-\psi_0^2=0. \tag{46}$$

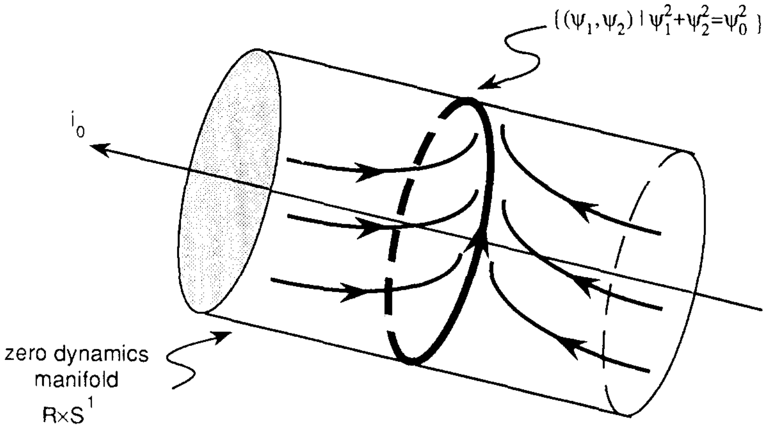

Fig. 2. The flow on the zero dynamics manifold. The attractor is a periodic orbit with frequency $\delta\omega$. If the load torque is zero, than $\delta\omega$ is zero and the attractor consists of a circle of equilibrium points.

From the defining relations for ψ_1, ψ_2 and (45) we obtain the stator currents

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \frac{L_f}{L_{fd}\psi_0}\begin{bmatrix} \psi_0/L_f & \tau \\ -\tau & \psi_0/L_f \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \tag{47}$$

Combining the last two equations of (44b) with (45) we obtain the flux differential equation

$$\frac{d}{dt}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = -\frac{r_f\tau}{\psi_0}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{48}$$

where initial conditions must satisfy (46). The general solution of (48) is

$$\psi_1 = \psi_0 \sin(\delta\omega t + \gamma), \quad \psi_2 = \psi_0 \cos(\delta\omega t + \gamma) \quad \text{and} \quad \delta\omega := \frac{r_f\tau}{\psi_0}. \tag{49}$$

Thus, we find that the flux is periodic with amplitude ψ_0 as required and that it has frequency $\delta\omega$. It follows from (45) and (47) that the stator and rotor currents are also periodic with this frequency. Note that $\delta\omega$ is the relative frequency of rotation as seen in the rotor fixed frame of reference. It is proportional to the load torque. Also observe that from (45) and (47) we can conclude that the rotor (field) current phasor leads the stator (d–q axis) current phasor by an angle of $\tan(\tau L_f/\psi_0)$. Finally, (45), (47) and (48) can be inserted into the first two relations of (44b) to obtain the effective driving voltage

$$\begin{bmatrix} v_d \\ v_q \end{bmatrix} = \left\{ R\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\delta\omega + S \right\}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{50}$$

where

$$R = \frac{L_s}{\sigma L_{fd}}\begin{bmatrix} \sigma & 1 \\ -1 & \delta \end{bmatrix}, \qquad S = \frac{1}{\sigma L_{fd}}\left\{ r\begin{bmatrix} \sigma & 1 \\ -1 & \sigma \end{bmatrix} + \omega_0 L_s\begin{bmatrix} \varepsilon & 0 \\ -\sigma & \varepsilon \end{bmatrix} \right\},$$

$$\sigma = \frac{1}{\delta\omega}\left[\frac{r_f}{(L_{fd}^2/L_f L_s) + 1} \right], \qquad \varepsilon = \left[\frac{(L_{fd}^2/L_f L_s) - 1}{(L_{fd}^2/L_f L_s) + 1} \right].$$

The matrix multiplier in (50) is a function of the relative frequency $\delta\omega$ and may be interpreted as a frequency transfer matrix. Thus, we see that the effective driving voltage along trajectories in the zero dynamics manifold is periodic with amplitude and phase (relative to the rotor flux) obtained from (50).

Finally, the design can be completed by selecting switching surfaces and establishing the reaching controls via the quadratic Liapunov function method as described in Section 3. It is worth noting that exponential speed decay to its desired value is accomplished by selecting the switching surfaces in the z coordinates as contrasted with incorporating this requirement into the definition of the regulated output as proposed in [12,17]. Similar remarks hold for the rotor field flux.

5. Concluding remarks

Many systems of interest to us are inherently nonlinear and require the design of nonlinear switching surfaces. Because of this, we have found it convenient to view the VS design process in the recently developed framework of partial linearization and zero dynamics for affine nonlinear systems. From this perspective the essential step in the design process is that of selecting the regulated outputs so that the zero dynamics have the required stability properties. Once this is done, the linearizable dynamics (11) can be computed. Equations (11) represent a convenient regular form for the design of a switching controller. A switching surface linear in the z variables results in an essentially standard VS control problem with linear plant because the equivalent control behaves as a linearizing control. Such a switching surface is easily obtained which will stabilize the sliding dynamics and we have given one simple construction in (27). Finally, the control functions $u_i^-(x)$ are chosen to stabilize the switching manifold. This element of the design process also benefits from the convenient structure of (11).

Note that the crucial step in the design is precisely the selection of a set of outputs y such that the zero dynamics are satisfactory. In many cases this means that the zero dynamics have a bounded, stable attractor. Of course, this corresponds to the selection of switching surfaces in the conventional VS formulation which result in a stable sliding mode. In the case of linear dynamics, switching surface design can be nicely addressed via techniques of linear system stabilization and a comprehensive theory is available (Young et al. [20], Kwatny and Young [15]). With nonlinear dynamics the available methods are ad hoc and rely principally on the ingenuity of the designer. Perhaps the best suggestion is that of Izosimov and Utkin [12] in which the desired dynamics are specified in terms of linear differential equations and the nonlinear switching surfaces derived therefrom. Typically, these differential equations are formulated in terms of the regulated variables. Notwithstanding some successful applications, such an approach suffers from several subtle difficulties. For instance the number of switching surfaces which may be consistently characterized by differential equations is not arbitrary and has not previously been identified.

The formulation outlined herein may be seen as enhancing this approach in several respects. First, the m switching surfaces defined in Proposition 2 can be interpreted as a set of algebraic and differential equations in the regulated outputs because z may be viewed as composed of the variable y and some of its time derivatives. These surfaces result in $r - m$ independent linear sliding differential equations – exactly the right number. Second, the essential nonlinear core of the sliding dynamics is isolated as the $n - r$ dimensional zero dynamics, which may be substantially smaller than the dimension of the sliding dynamics $(n - m)$ as pointed out in [4]. Indeed, we have seen that the zero dynamics manifold belongs to the intersection of the output constraint manifold and the sliding manifold. Finally, the approach given here is systematic.

It is well known that the sliding mode behavior is precisely the zero dynamics with respect to the sliding surfaces $s(x)$ provided that the condition (4) holds, cf. Young et al [20] for the case of linear dynamics and Marino [16] for the case of linearizable dynamics. Byrnes and Isidori [4] suggest how to construct switching surfaces provided the system is minimum phase at an equilibrium point with respect to the regulated outputs $y = h(x)$ with the result that the system zero dynamics is a subset of the sliding dynamics. In fact, this is precisely the correct theoretical framework for systematizing the method of Izosimov and Utkin [12].

Although the identification of the linearizable dynamics (11) is straightforward, the direct identification of the zero dynamics in local explicit form is not. On the other hand, the zero dynamics may be analyzed by investigating the constrained dynamics (14). This is one reason that we prefer to deal with the zero dynamics in the implicit (constrained) form of (14). We emphasize, however, that in some important applications, such as the induction motor described in Section 4, a local characterization of the zero dynamics is not appropriate, underscoring the importance of (14).

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